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From the quantum Jacobi–Trudi and Giambelli formula to a nonlinear integral equation for thermodynamics of the higher spin Heisenberg model

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Abstract

We propose a nonlinear integral equation (NLIE) with only one unknown function, which gives the free energy of the integrable one-dimensional Heisenberg model with arbitrary spin. In deriving the NLIE, the quantum Jacobi–Trudi and Giambelli formula (Bazhanov–Reshetikhin formula), which gives the solution of the T-system, plays an important role. In addition, we also calculate the high temperature expansion of the specific heat and the magnetic susceptibility.

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1. Introduction

Thermodynamic Bethe Ansatz (TBA) equations have been used to analyse thermodynamics of various kinds of solvable lattice models [1]. However, it is not always easy to treat TBA equations since in general they contain an infinite number of unknown functions. So there are several attempts to simplify the TBA equations. In particular for the 1D spin-$\frac{1}{2}$ Heisenberg model, Klümper proposed nonlinear integral equations (NLIE) [2] which contain a finite number of unknown functions from the point of view of the quantum transfer matrix (QTM) approach [2–7]. There is also a similar NLIE by Destri and de Vega [8].

Another type of NLIE for the spin-$\frac{1}{2}XXZ$ model, which contains only one unknown function, was discovered by Takahashi [9] to simplify the TBA equation. Later, this NLIE was rederived [10] from the T-system [11, 12] of the QTM. Moreover this NLIE for the spin-$\frac{1}{2}XXX$ model was also rederived from a fugacity expansion formula [13]. We derived this type of NLIE for the $osp(1|2s)$ model [14] and the $sl(r+1)$ Uimin–Sutherland model [15] for *arbitrary* rank. The number of unknown functions of these NLIE coincides with the rank of

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the underlying algebras. All of them are NLIE for fundamental representations of underlying algebras. So it is desirable to derive NLIE for tensor representations, i.e. NLIE for higher spin models. The purpose of this paper is to derive NLIE for the Heisenberg spin chain with *arbitrary* spin $\frac{s}{2}$ [16].

Thermodynamics of the higher spin Heisenberg model was first investigated [16] by the TBA equations, which contain an infinite number of unknown functions. Later a set of NLIE with $s+1$ unknown functions for this model was derived [17] using the QTM approach. This NLIE is an extension of Klümper's type of NLIE [2]. On the other hand, our new NLIE contains only *one* unknown function for *arbitrary* s. Thus, as far as the number of unknown functions is concerned, our new NLIE is a further simplification of the TBA equations.

In section 2, we introduce the higher spin Heisenberg model and define the T-system of the QTM. As a solution of the T-system, we introduce the quantum Jacobi–Trudi and Giambelli formula (Bazhanov–Reshetikhin formula [18]) (2.18), which plays an essential role in the derivation of the NLIE. This formula expresses an eigenvalue formula of the transfer matrix for the tensor representation in the auxiliary space in a determinant form. In the representation theoretical context, it may be viewed as a Yangian analogue of classical Jacobi–Trudi and Giambelli formula on Schur functions [19, 18]. In section 3, we derive the NLIE (3.9), which is our main result. The T-system, which we have to use here, is not the standard one [12] (2.15) but an old one [11] (2.20), (2.21). Due to the quantum Jacobi–Trudi and Giambelli formula, determinants appear in our new NLIE. These novel situations contrast with the fundamental representation cases [9, 10, 14, 15]. Using our new NLIE (3.9), we also calculate the high temperature expansion of the specific heat and the magnetic susceptibility in section 4. It will be difficult to obtain the same result using the traditional TBA equations. Section 5 is devoted to concluding remarks.

2. QTM method, *T*-system, quantum Jacobi–Trudi and Giambelli formula

We introduce the higher spin Heisenberg model, and define the QTM, the T-system and the quantum Jacobi–Trudi and Giambelli formula for this model. A more detailed explanation of the QTM analysis for this model can be found in [17].

The Hamiltonian of the spin-$\frac{s}{2}$ Heisenberg model is given as follows [16]:

$$\mathcal{H}_0 = J\sum_{j=1}^{L}\mathcal{Q}_s(\mathbf{S}_j\mathbf{S}_{j+1}) \tag{2.1}$$

where $\mathbf{S}_j = \left(S_j^x, S_j^y, S_j^z\right)$ is the spin $\frac{s}{2}$ operator acting on the jth site, and $\mathbf{S}_j\mathbf{S}_{j+1} = S_j^xS_{j+1}^x + S_j^yS_{j+1}^y + S_j^zS_{j+1}^z$. J is a real coupling constant ($J > 0$ (respectively $J < 0$) corresponds to the anti-ferromagnetic (respectively ferromagnetic) regime) and L is the number of lattice sites. We assume the periodic boundary condition $\mathbf{S}_{L+1} = \mathbf{S}_1 \cdot \mathcal{Q}_s(x)$ is defined as

$$\mathcal{Q}_s(x) = -2\sum_{j=0}^{s-1}\sum_{k=j+1}^{s}\frac{1}{k}\prod_{p=0(p\neq j)}^{s}\frac{x-x_p}{x_j-x_p} \tag{2.2}$$

where $x_p = \frac{1}{2}\left(p(p+1) - s\left(\frac{s}{2}+1\right)\right)$. For the $s = 1$ and $s = 2$ case, (2.1) becomes

$$\mathcal{H}_0 = 2J\sum_{j=1}^{L}\left\{\mathbf{S}_j\mathbf{S}_{j+1} - \frac{1}{4}\right\} \qquad \mathcal{H}_0 = \frac{J}{2}\sum_{j=1}^{L}\{\mathbf{S}_j\mathbf{S}_{j+1} - (\mathbf{S}_j\mathbf{S}_{j+1})^2\} \tag{2.3}$$

respectively. The R-matrix [20, 21] of the classical counterpart of the spin $\frac{s}{2}$ Heisenberg model is defined as

$$R(v) = \sum_{j=0}^{s} \left(\prod_{k=j+1}^{s} \frac{v-k}{v+k} \right) P^j \tag{2.4}$$

where P^j is the projector onto $(j+1)$-dimensional irreducible $sl(2)$ module. P^j can be expressed as

$$P^j = \prod_{p=0(p\neq j)}^{s} \frac{\mathbf{S}\otimes\mathbf{S} - x_p}{x_j - x_p} \tag{2.5}$$

where $\mathbf{S} = (S^x, S^y, S^z)$ is the spin $\frac{s}{2}$ operator, and $\mathbf{S}\otimes\mathbf{S} = S^x \otimes S^x + S^y \otimes S^y + S^z \otimes S^z$. In our normalization of the R-matrix, $R(\infty)$ is an identity operator. The row to row transfer matrix is defined as

$$t(v) = tr_0 R_{0L}(v) \cdots R_{02}(v) R_{01}(v) \tag{2.6}$$

where $R_{0j}(v)$ is the R-matrix (2.4) acting on the auxiliary space and the jth site of the quantum space. This transfer matrix is connected to the Hamiltonian (2.1) as

$$\mathcal{H}_0 = J \frac{\mathrm{d}}{\mathrm{d}v} \log t(v)|_{v=0}. \tag{2.7}$$

One can add an external field term $\mathcal{H}_{\mathrm{ex}} = -2h \sum_j S_j^z$ in the Hamiltonian (2.1) without breaking the integrability as $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\mathrm{ex}}$. The QTM is defined as

$$t_{\mathrm{QTM}}(v) = tr_0\, \mathrm{e}^{\frac{2hS_0^z}{T}} \tilde{R}_{N0}(u - \mathrm{i}v) R_{N-10}(u + \mathrm{i}v) \cdots \tilde{R}_{20}(u - \mathrm{i}v) R_{10}(u + \mathrm{i}v) \tag{2.8}$$

where $\tilde{R}_{jk}(v)$ is defined by '90° rotation' of $R_{kj}(v)$, i.e., $\tilde{R}_{jk}(v) = {}^{t_k}R_{kj}(v)$ (t_k: the transposition of $R_{kj}(v)$ in the kth space); N is the Trotter number and assumed to be even; $u = -\frac{J}{TN}$ (T is temperature); the Boltzmann constant is set to 1. The free energy per site is expressed in terms of the largest eigenvalue $\tilde{T}_s(v)$ of the QTM (2.8) at $v = 0$:

$$f = -T \lim_{N\to\infty} \log \tilde{T}_s(0). \tag{2.9}$$

We can embed $\tilde{T}_s(v)$ into the eigenvalue formulae $\{\tilde{T}_m(v)\}_{m\in\mathbb{Z}_{\geqslant 0}}$ for the fusion hierarchy of the QTM, i.e. a Yang–Baxterization of the character of the mth symmetric tensor representation of $sl(2)$:

$$\tilde{T}_m(v) = T_m(v)/\mathcal{N}_m(v) \tag{2.10}$$

where

$$\begin{aligned} T_m(v) = & \sum_{1\leqslant d_1\leqslant d_2\leqslant\cdots\leqslant d_m\leqslant 2} \prod_{j=1}^{m} z\left(d_j; v + \frac{\mathrm{i}}{2}(m-2j+1)\right) \\ = & \sum_{k=0}^{m} \mathrm{e}^{\frac{(m-k)\mu_1 + k\mu_2}{T}} \prod_{j=1}^{m-k} \phi_-\left(v + \frac{m-s-2j}{2}\mathrm{i}\right) \phi_+\left(v + \frac{m-s-2j+2}{2}\mathrm{i}\right) \\ & \times \prod_{j=m-k+1}^{m} \phi_-\left(v + \frac{m+s-2j}{2}\mathrm{i}\right) \phi_+\left(v + \frac{m+s-2j+2}{2}\mathrm{i}\right) \\ & \times \frac{Q\left(v + \frac{m}{2}\mathrm{i}\right) Q\left(v - \frac{m+2}{2}\mathrm{i}\right)}{Q\left(v - \frac{m-2k}{2}\mathrm{i}\right) Q\left(v - \frac{m-2k+2}{2}\mathrm{i}\right)} \end{aligned} \tag{2.11}$$

$$\mathcal{N}_m(v) = \prod_{k=1}^{m} \phi_-\left(v + \frac{m-2k-s}{2}\mathrm{i}\right) \phi_+\left(v + \frac{m-2k+s+2}{2}\mathrm{i}\right) \tag{2.12}$$

$\phi_\pm(v) = (v \pm \mathrm{i}u)^{\frac{N}{2}}$; $Q(v) = \prod_{j=1}^{M}(v - v_j)$ ($M \in \mathbb{Z}_{\geqslant 0}$) and μ_1 and μ_2 are chemical potentials (in our case, $\mu_1 = h$, $\mu_2 = -h$);

$$\begin{aligned} z(1; v) &= \mathrm{e}^{\frac{\mu_1}{T}} \phi_-\left(v - \frac{s+1}{2}\mathrm{i}\right) \phi_+\left(v - \frac{s-1}{2}\mathrm{i}\right) \frac{Q\left(v + \frac{\mathrm{i}}{2}\right)}{Q\left(v - \frac{\mathrm{i}}{2}\right)} \\ z(2; v) &= \mathrm{e}^{\frac{\mu_2}{T}} \phi_-\left(v + \frac{s-1}{2}\mathrm{i}\right) \phi_+\left(v + \frac{s+1}{2}\mathrm{i}\right) \frac{Q\left(v - \frac{3\mathrm{i}}{2}\right)}{Q\left(v - \frac{\mathrm{i}}{2}\right)}. \end{aligned} \tag{2.13}$$

Here we adopt a convention $\prod_{j=1}^{0}(\cdots) = \prod_{j=m+1}^{m}(\cdots) = 1$. v_j is a solution of the Bethe Ansatz equation

$$-\frac{\phi_-\left(v_k - \frac{s\mathrm{i}}{2}\right)\phi_+\left(v_k - \frac{s\mathrm{i}}{2} + \frac{g\mathrm{i}}{2}\right)}{\phi_-\left(v_k + \frac{s\mathrm{i}}{2}\right)\phi_+\left(v_k + \frac{s\mathrm{i}}{2} + \frac{g\mathrm{i}}{2}\right)} = \mathrm{e}^{-\frac{\mu_1-\mu_2}{T}} \frac{Q(v_k - \mathrm{i})}{Q(v_k + \mathrm{i})} \qquad k \in \{1, 2, \ldots, M\} \tag{2.14}$$

where $g = 2$: the dual Coxeter number. Note that $T_m(v)$ (2.11) is free of poles under the Bethe Ansatz equation (2.14). One can show that the function $\tilde{T}_m(v)$ (2.10) satisfies the following T-system [12]:

$$\tilde{T}_m\left(v - \frac{\mathrm{i}}{2}\right)\tilde{T}_m\left(v + \frac{\mathrm{i}}{2}\right) = \tilde{T}_{m-1}(v)\tilde{T}_{m+1}(v) + \tilde{g}_m(v) \qquad m \in \mathbb{Z}_{\geqslant 1} \tag{2.15}$$

where

$$\tilde{T}_0(v) = 1 \tag{2.16}$$

$$\tilde{g}_m(v) = \mathrm{e}^{\frac{m(\mu_1+\mu_2)}{T}} \prod_{k=1}^{\min(m,s)} \frac{\phi_-\left(v + \frac{m+s+1-2k}{2}\mathrm{i}\right)\phi_+\left(v - \frac{m+s+1-2k}{2}\mathrm{i}\right)}{\phi_-\left(v - \frac{m+s+1-2k}{2}\mathrm{i}\right)\phi_+\left(v + \frac{m+s+1-2k}{2}\mathrm{i}\right)}. \tag{2.17}$$

The solution of the T-system (2.15) is given by the following quantum Jacobi–Trudi and Giambelli formula (Bazhanov–Reshetikhin formula [18]):

$$\tilde{T}_m(v) = \det_{1\leqslant j,k\leqslant m}\left(\tilde{f}^{1+j-k}\left(v - \frac{j+k-m-1}{2}\mathrm{i}\right)\right) \tag{2.18}$$

where the matrix elements are given as follows

$$\tilde{f}^a(v) = \begin{cases} 1 & a = 0 \\ \tilde{T}_1(v) & a = 1 \\ \tilde{g}_1(v) & a = 2 \\ 0 & a > 2 \quad \text{or} \quad a < 0. \end{cases} \tag{2.19}$$

The following functional relations [11] are equivalent to the T-system (2.15):

$$\tilde{T}_1(v)\tilde{T}_{m-1}\left(v - \frac{m}{2}\mathrm{i}\right) = \tilde{T}_m\left(v - \frac{m-1}{2}\mathrm{i}\right) + \tilde{g}_1\left(v - \frac{\mathrm{i}}{2}\right)\tilde{T}_{m-2}\left(v - \frac{m+1}{2}\mathrm{i}\right) \tag{2.20}$$

$$\tilde{T}_1(v)\tilde{T}_{m-1}\left(v + \frac{m}{2}\mathrm{i}\right) = \tilde{T}_m\left(v + \frac{m-1}{2}\mathrm{i}\right) + \tilde{g}_1\left(v + \frac{\mathrm{i}}{2}\right)\tilde{T}_{m-2}\left(v + \frac{m+1}{2}\mathrm{i}\right) \qquad m \in \mathbb{Z}_{\geqslant 1}. \tag{2.21}$$

(2.20) follows from an expansion of the determinant (2.18) down the first row. (2.21) follows from an expansion of the determinant (2.18) down the last column.

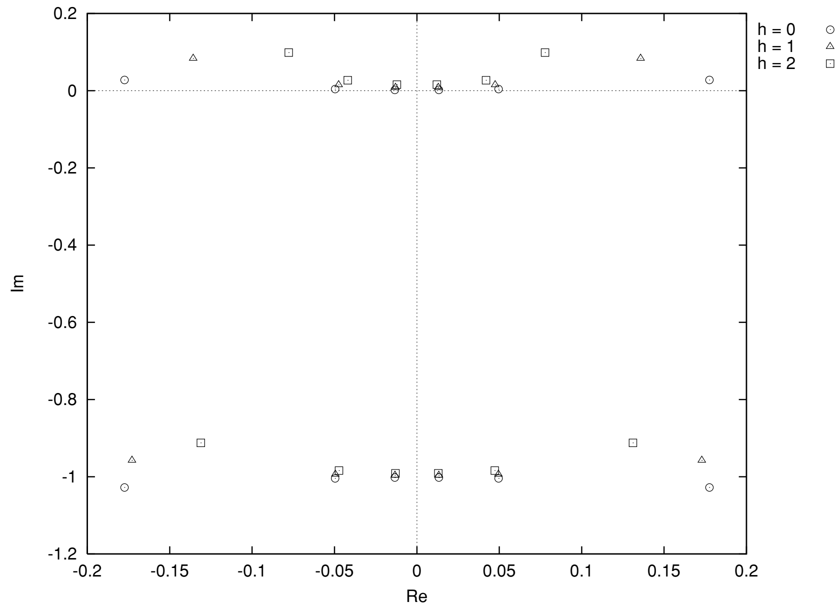

Figure 1. Location of the roots of the BAE (2.14) for the spin $\frac{s}{2} = 1$ case ($N = 12, u = -0.05, J = 1, h = 0, 1, 2$), which will give the largest eigenvalue for $\tilde{T}_2(v)$ at $v = 0$. The root forms 2-strings at least for $h = 0$.

3. A new nonlinear integral equation

Suzuki [17] had the following observation from numerics.

For the largest eigenvalue sector of $\tilde{T}_s(v)$, the corresponding root of the BAE (2.14) forms s-strings. In this case, imaginary parts of the zeros $\{\tilde{z}_m\}$ of $\tilde{T}_m(v)$ $(m = 1, 2, \ldots, s)$ are located near the lines $\mathrm{Im} v = \pm\frac{1}{2}(s + m - 2j)$, $j = 0, 1, \ldots, m - 1$.

Following Suzuki's calculations for the spin $\frac{s}{2} = 1$ case, we have plotted roots of the BAE (2.14) in the sector $M = N$ and zeros of $\tilde{T}_m(v)$ $(m = 1, 2)$ (figures 1 and 2). Admitting Suzuki's observation, we shall derive the NLIE for the largest eigenvalue sector of $\tilde{T}_s(v)$ from now on. $\tilde{T}_m(v)$ has poles[1] at $v = \pm\tilde{\beta}_m : \tilde{\beta}_m = \left(\frac{m+s}{2} + u\right)\mathrm{i}, \left(\frac{m+s-2}{2} + u\right)\mathrm{i}, \ldots, \left(\frac{m+s+2-2\min(m,s)}{2} + u\right)\mathrm{i}$, whose order is at most $\frac{N}{2}$. Moreover

$$Q_m := \lim_{|v|\to\infty} \tilde{T}_m(v) = \sum_{k=0}^{m} \mathrm{e}^{\frac{(m-k)\mu_1 + k\mu_2}{T}} = \sum_{k=0}^{m} \mathrm{e}^{\frac{h(m-2k)}{T}} \tag{3.1}$$

is a finite number. This is a solution of the Q-system [22, 23]

$$(Q_m)^2 = Q_{m-1}Q_{m+1} + Q_m^{(2)} \qquad m \in \mathbb{Z}_{\geqslant 1} \tag{3.2}$$

where $Q_m^{(2)} = \mathrm{e}^{\frac{m(\mu_1+\mu_2)}{T}} = 1$ and $Q_0 = 1$. So we must put

$$\tilde{T}_1(v) = Q_1 + \sum_{j=1}^{\frac{N}{2}} \left\{ \frac{b_j}{(v - \tilde{\beta}_1)^j} + \frac{\bar{b}_j}{(v + \tilde{\beta}_1)^j} \right\} \tag{3.3}$$

[1] Note that these poles are known ones, which we need not solve the BAE (2.14) to obtain.

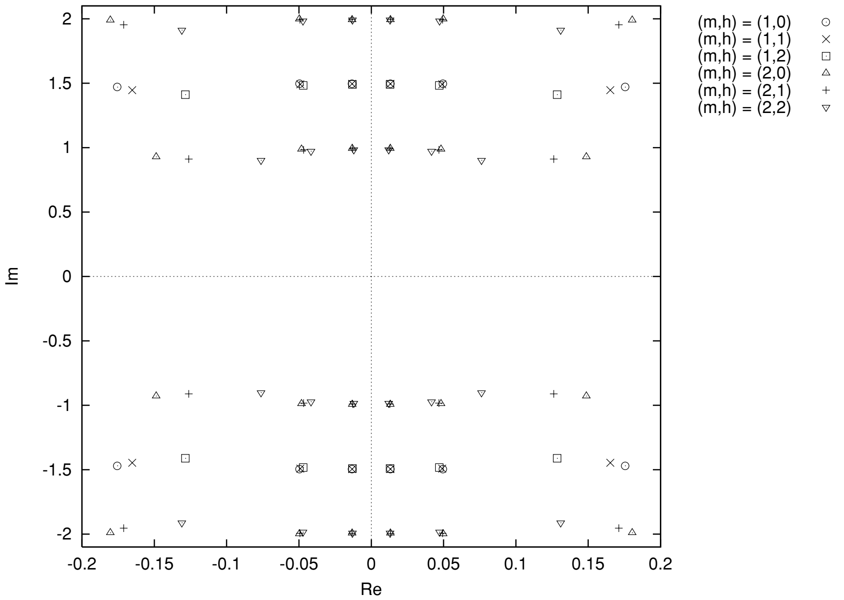

Figure 2. Location of the zeros of $\tilde{T}_m(v)$ $(m = 1, 2)$ for the roots in figure 1. The zeros are located outside the physical strip $\operatorname{Im} v \in [-\frac{1}{2}, \frac{1}{2}]$.

where $\tilde{\beta}_1 = \left(\frac{1+s}{2} + u\right)\mathrm{i}$. Here the coefficients are given as follows:

$$b_j = \oint_C \frac{\mathrm{d}v}{2\pi\mathrm{i}} \tilde{T}_1(v)(v - \tilde{\beta}_1)^{j-1} \qquad \bar{b}_j = \oint_{\bar{C}} \frac{\mathrm{d}v}{2\pi\mathrm{i}} \tilde{T}_1(v)(v + \tilde{\beta}_1)^{j-1} \tag{3.4}$$

where the contour C (respectively $\bar{C}$) is an anti-clockwise closed loop around $\tilde{\beta}_1$ (respectively $-\tilde{\beta}_1$) that does not surround $-\tilde{\beta}_1$ (respectively $\tilde{\beta}_1$). Substituting (3.4) into (3.3), and using (2.20) and (2.21), we obtain

$$\begin{aligned}\tilde{T}_1(v) = Q_1 &+ \oint_C \frac{\mathrm{d}y}{2\pi\mathrm{i}} \frac{1 - \left(\frac{y}{v-\tilde{\beta}_1}\right)^{\frac{N}{2}}}{v - y - \tilde{\beta}_1} \left\{ \frac{\tilde{T}_m\left(y + \tilde{\beta}_1 - \frac{m-1}{2}\mathrm{i}\right)}{\tilde{T}_{m-1}\left(y + \tilde{\beta}_1 - \frac{m}{2}\mathrm{i}\right)} \right. \\ &\left. + \frac{\tilde{g}_1\left(y + \tilde{\beta}_1 - \frac{\mathrm{i}}{2}\right)\tilde{T}_{m-2}\left(y + \tilde{\beta}_1 - \frac{m+1}{2}\mathrm{i}\right)}{\tilde{T}_{m-1}\left(y + \tilde{\beta}_1 - \frac{m}{2}\mathrm{i}\right)} \right\} \\ &+ \oint_{\bar{C}} \frac{\mathrm{d}y}{2\pi\mathrm{i}} \frac{1 - \left(\frac{y}{v+\tilde{\beta}_1}\right)^{\frac{N}{2}}}{v - y + \tilde{\beta}_1} \left\{ \frac{\tilde{T}_m\left(y - \tilde{\beta}_1 + \frac{m-1}{2}\mathrm{i}\right)}{\tilde{T}_{m-1}\left(y - \tilde{\beta}_1 + \frac{m}{2}\mathrm{i}\right)} \right. \\ &\left. + \frac{\tilde{g}_1\left(y - \tilde{\beta}_1 + \frac{\mathrm{i}}{2}\right)\tilde{T}_{m-2}\left(y - \tilde{\beta}_1 + \frac{m+1}{2}\mathrm{i}\right)}{\tilde{T}_{m-1}\left(y - \tilde{\beta}_1 + \frac{m}{2}\mathrm{i}\right)} \right\}\end{aligned} \tag{3.5}$$

where the contour C (respectively $\bar{C}$) is an anti-clockwise closed loop around 0 that does not surround $-2\tilde{\beta}_1$ (respectively $2\tilde{\beta}_1$). The first term and the second term in the first bracket $\{\cdots\}$ in (3.5) have a common pole at 0. However, this common pole of the first term disappears if $m \geqslant s + 1$. Therefore, for $m \geqslant s + 1$ the first term vanishes after the integration as long as the contour C does not surround the pole at $\tilde{z}_{m-1} - \tilde{\beta}_1 + \frac{m}{2}\mathrm{i}$. Similarly, for $m \geqslant s + 1$, the first term in the second bracket $\{\cdots\}$ vanishes after the integration as long as the contour $\bar{C}$ does not surround the pole at $\tilde{z}_{m-1} + \tilde{\beta}_1 - \frac{m}{2}\mathrm{i}$. Thus, for $m \geqslant s+1$, we obtain

$$\tilde{T}_1(v) = Q_1 + \oint_C \frac{\mathrm{d}y}{2\pi\mathrm{i}} \frac{\tilde{g}_1\left(y + \tilde{\beta}_1 - \frac{\mathrm{i}}{2}\right)\tilde{T}_{m-2}\left(y + \tilde{\beta}_1 - \frac{m+1}{2}\mathrm{i}\right)}{(v - y - \tilde{\beta}_1)\tilde{T}_{m-1}\left(y + \tilde{\beta}_1 - \frac{m}{2}\mathrm{i}\right)} + \oint_{\bar{C}} \frac{\mathrm{d}y}{2\pi\mathrm{i}} \frac{\tilde{g}_1\left(y - \tilde{\beta}_1 + \frac{\mathrm{i}}{2}\right)\tilde{T}_{m-2}\left(y - \tilde{\beta}_1 + \frac{m+1}{2}\mathrm{i}\right)}{(v - y + \tilde{\beta}_1)\tilde{T}_{m-1}\left(y - \tilde{\beta}_1 + \frac{m}{2}\mathrm{i}\right)}. \tag{3.6}$$

Here we omit the terms which contain $y^{N/2}$ since these terms cancel the poles of $\tilde{g}_1$. We can take the Trotter limit $N \to \infty$,

$$\mathcal{T}_1(v) = Q_1 + \oint_C \frac{\mathrm{d}y}{2\pi\mathrm{i}} \frac{g_1\left(y + \beta_1 - \frac{\mathrm{i}}{2}\right)\mathcal{T}_{m-2}\left(y + \beta_1 - \frac{m+1}{2}\mathrm{i}\right)}{(v - y - \beta_1)\mathcal{T}_{m-1}\left(y + \beta_1 - \frac{m}{2}\mathrm{i}\right)} + \oint_{\bar{C}} \frac{\mathrm{d}y}{2\pi\mathrm{i}} \frac{g_1\left(y - \beta_1 + \frac{\mathrm{i}}{2}\right)\mathcal{T}_{m-2}\left(y - \beta_1 + \frac{m+1}{2}\mathrm{i}\right)}{(v - y + \beta_1)\mathcal{T}_{m-1}\left(y - \beta_1 + \frac{m}{2}\mathrm{i}\right)} \tag{3.7}$$

where $\beta_1 = \lim_{N\to\infty} \tilde{\beta}_1 = \frac{1+s}{2}\mathrm{i}$, $\mathcal{T}_m(v) = \lim_{N\to\infty} \tilde{T}_m(v)$ and

$$g_1(v) = \exp\left(\frac{1}{T}\left\{\frac{Js}{\left(v^2 + \frac{s^2}{4}\right)} + \mu_1 + \mu_2\right\}\right) = Q_1^{(2)} \exp\left(\frac{Js}{\left(v^2 + \frac{s^2}{4}\right)T}\right). \tag{3.8}$$

In particular, (3.7) for $m = s+1$ is the simplest,

$$\mathcal{T}_1(v) = Q_1 + \oint_C \frac{\mathrm{d}y}{2\pi\mathrm{i}} \frac{g_1\left(y + \frac{s}{2}\mathrm{i}\right)\mathcal{T}_{s-1}\left(y - \frac{\mathrm{i}}{2}\right)}{\left(v - y - \frac{s+1}{2}\mathrm{i}\right)\mathcal{T}_s(y)} + \oint_{\bar{C}} \frac{\mathrm{d}y}{2\pi\mathrm{i}} \frac{g_1\left(y - \frac{s}{2}\mathrm{i}\right)\mathcal{T}_{s-1}\left(y + \frac{\mathrm{i}}{2}\right)}{\left(v - y + \frac{s+1}{2}\mathrm{i}\right)\mathcal{T}_s(y)} \tag{3.9}$$

where the contour C (respectively $\bar{C}$) is an anti-clockwise closed loop around 0 that does not surround $-(1+s)\,\mathrm{i}$ (respectively $(1+s)\mathrm{i}$) and z_s. (3.9) contains only one unknown function $\mathcal{T}_1(v)$ since $\mathcal{T}_s(v)$ and $\mathcal{T}_{s-1}(v)$ can be expressed by $\mathcal{T}_1(v)$ through (2.18) in the Trotter limit. The free energy per site is given by $f = -T \log \mathcal{T}_s(0)$. For $s = 1$, (3.9) reduces to Takahashi's NLIE for the *XXX* spin chain [9]. Although we only consider the largest eigenvalue of $\tilde{T}_s(v)$ in the limit $N \to \infty$, other eigenvalues also satisfy the NLIE (3.9) if above conditions for the integral contours are satisfied. As usual, such eigenvalues have zeros in the physical strip $\operatorname{Im} v \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Thus to exclude the eigenvalues other than the largest one, one may take the integral contours on the line $\operatorname{Im} v = \pm\frac{1}{2}$.

4. High temperature expansion

In this section, we shall calculate the high temperature expansion of the free energy from our new NLIE (3.9). The calculation is not easier than the $s = 1$ case [24] due to the determinants in (3.9). However it is easier than using the traditional TBA equation. For small J/T, we shall put

$$\mathcal{T}_m(v) = \exp\left(\sum_{n=0}^{\infty} b_{m,n}(v)\left(\frac{J}{T}\right)^n\right) \tag{4.1}$$

where $b_{m,0}(v) = \log Q_m$. Due to (2.18) in the limit $N \to \infty$, one can express $b_{m,n}(v)$ in terms of fundamental ones $\{b_{1,k}(v)\}_{0\leqslant k\leqslant n}$, $Q_1^{(2)}$ and $b(v) = s\big/\left(v^2 + \frac{s^2}{4}\right)$. For example, we have

$$b_{2,1}(v) = \left((Q_1)^2 b_{1,1}\left(v - \frac{\mathrm{i}}{2}\right) + (Q_1)^2 b_{1,1}\left(v + \frac{\mathrm{i}}{2}\right) - Q_1^{(2)} b(v)\right)\Big/\, Q_2 \tag{4.2}$$

where $Q_2 = (Q_1)^2 - Q_1^{(2)}$. Taking note of relations such as (4.2) and substituting (4.1) into (3.9), we obtain the coefficients $\{b_{m,n}(v)\}$. For example, $\{b_{1,n}(v)\}$ for $s = 2, n = 1, 2, 3$ are as follows:

$$\begin{aligned}
b_{1,1}(v) &= \frac{12Q_1^{(2)}}{(9+4v^2)\big((Q_1)^2 - Q_1^{(2)}\big)} \\
b_{1,2}(v) &= 2Q_1^{(2)}\big\{(45+4v^2)(Q_1)^4 + (-99+4v^2)(Q_1)^2 Q_1^{(2)} \\
&\quad - 4(27+20v^2)\big(Q_1^{(2)}\big)^2\big\}\big/\big\{(9+4v^2)^2\big((Q_1)^2 - Q_1^{(2)}\big)^3\big\} \\
b_{1,3}(v) &= \big\{Q_1^{(2)}\big((477+120v^2+16v^4)(Q_1)^8 + 9(-179+56v^2+16v^4)(Q_1)^6 Q_1^{(2)} \\
&\quad - 6(1101+1536v^2+304v^4)(Q_1)^4\big(Q_1^{(2)}\big)^2 \\
&\quad + 4(5553+5028v^2+896v^4)(Q_1)^2\big(Q_1^{(2)}\big)^3 + 48(63+84v^2 \\
&\quad + 32v^4)\big(Q_1^{(2)}\big)^4\big)\big\}\big/\big\{(9+4v^2)^3\big((Q_1)^2 - Q_1^{(2)}\big)^5\big\}.
\end{aligned} \tag{4.3}$$

We can calculate the specific heat $C = -T\frac{\partial^2 f}{\partial T^2}$ and the magnetic susceptibility $\chi = -\frac{\partial^2 f}{\partial h^2}\big|_{h=0}$. In this case, we only use $\{b_{m,k}(v)\}$ for $v = 0$ due to the definition of the free energy (2.9). Note that the h-dependence of the free energy enters only through $Q_1 = \mathrm{e}^{\frac{h}{T}} + \mathrm{e}^{-\frac{h}{T}}$ since $Q_1^{(2)} = 1$. Let us put $t = J/T$.

$s = 2$ case:

$$\begin{aligned}
C &= \frac{8t^2}{9} + \frac{34t^3}{27} - \frac{5t^4}{54} - \frac{580t^5}{243} - \frac{27629t^6}{11664} + \frac{165529t^7}{116640} + \frac{40875277t^8}{8398080} \\
&\quad + \frac{10648871t^9}{4408992} - \frac{2176205977t^{10}}{470292480} - \frac{94355582827t^{11}}{12697896960} - \frac{100181936647t^{12}}{304749527040} \\
&\quad + \frac{23030455724107t^{13}}{2394460569600} + \frac{4273471680238097t^{14}}{482723250831360} - \frac{919571868309546869t^{15}}{188262067824230400} \\
&\quad - \frac{30226224111769481353t^{16}}{1916850145119436800} - \frac{1741566568691332074437t^{17}}{237210205458530304000} \\
&\quad + \frac{636000568233936625686389t^{18}}{45544359448037818368000} + \frac{5325661218179817974957t^{19}}{248158368787385548800} \\
&\quad + \frac{143978522302549633307591t^{20}}{227535901895503223193600} - \frac{3868537325098539118572347677t^{21}}{144433584998668295995392000} \\
&\quad + O(t^{22})
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
\chi T &= \frac{8}{3} - \frac{8t}{3} + \frac{14t^3}{27} + \frac{205t^4}{324} + \frac{97t^5}{810} - \frac{32627t^6}{58320} - \frac{290839t^7}{489888} + \frac{2993083t^8}{26127360} \\
&\quad + \frac{51476713t^9}{70543872} + \frac{392473169t^{10}}{846526464} - \frac{6147244063t^{11}}{14549673600} - \frac{28552400626009t^{12}}{33522447974400} \\
&\quad - \frac{41436528439217t^{13}}{186767924428800} + \frac{46001003925515t^{14}}{59765735817216} + \frac{5761065533476745581t^{15}}{6589172373848064000} \\
&\quad - \frac{393055654682062161341t^{16}}{2530242191557656576000} - \frac{1142327068285920573167t^{17}}{1024145648963813376000} \\
&\quad + O(t^{18}).
\end{aligned} \tag{4.5}$$

$s = 3$ case:

$$\begin{aligned} C = {} & \frac{15t^2}{16} + \frac{145t^3}{96} + \frac{1385t^4}{2304} - \frac{13445t^5}{6912} - \frac{1203755t^6}{331776} - \frac{7458199t^7}{5971968} \\ & + \frac{66731599t^8}{15925248} + \frac{29819652545t^9}{4514807808} + \frac{548487273433t^{10}}{433421549568} - \frac{62192420939825t^{11}}{7801587892224} \\ & - \frac{2935934459029971 1t^{12}}{2808571641200640} - \frac{1535815314115199t^{13}}{4413469721886720} \\ & + \frac{1235553149703549929 5t^{14}}{889755495932362752} + \frac{534342962613181610732 3t^{15}}{347004643413621473280} + O(t^{16}) \end{aligned} \tag{4.6}$$

$$\begin{aligned} \chi T = {} & 5 - 5t + \frac{35t^3}{54} + \frac{2725t^4}{2592} + \frac{3383t^5}{5184} - \frac{173425t^6}{373248} - \frac{7164095t^7}{5878656} \\ & - \frac{9795049t^8}{13934592} + \frac{634647695t^9}{952342272} + \frac{2120381481515t^{10}}{1462797729792} + \frac{13785996387509t^{11}}{20113468784640} \\ & - \frac{33615348883175267t^{12}}{34756074059857920} - \frac{37654994203903963t^{13}}{21515664894197760} + O(t^{14}). \end{aligned} \tag{4.7}$$

$s = 4$ case:

$$\begin{aligned} C = {} & \frac{24t^2}{25} + \frac{619t^3}{375} + \frac{49159t^4}{45000} - \frac{900239t^5}{675000} - \frac{257362861t^6}{64800000} \\ & - \frac{190318307851t^7}{58320000000} + \frac{1061704692647t^8}{518400000000} + \frac{33531890711924393t^9}{4408992000000000} \\ & + \frac{13698973673330960069t^{10}}{2116316160000000000} - \frac{4091902458911705383t^{11}}{1360488960000000000} \\ & - \frac{17670440449523499058692 83t^{12}}{137137287168000000000000} + O(t^{13}) \end{aligned} \tag{4.8}$$

$$\begin{aligned} \chi T = {} & 8 - 8t + \frac{101t^3}{135} + \frac{9337t^4}{6480} + \frac{209797t^5}{162000} - \frac{7464779t^6}{116640000} \\ & - \frac{234988285877t^7}{146966400000} - \frac{3029100708947t^8}{1741824000000} - \frac{393143129330767t^9}{3809369088000000} \\ & + \frac{430378968817837183 1t^{10}}{2285621452800000000} + \frac{27858925530399279724 7t^{11}}{12570917990400000000 0} + O(t^{12}). \end{aligned} \tag{4.9}$$

We have plotted[2] the high temperature expansion of the specific heat and magnetic susceptibility in figures 3, 4 and 5. According to figure 3, the position of the peak of the specific heat does not seem to change drastically when s changes. In particular, $s = 2, 3$ cases agree with the result from another NLIE for large T (see figure 6 in [17]). This indicates the validity of our new NLIE.

[2] Here we have used the Pade approximation. The $[L, M]$ Pade approximation of a function $g(t)$ of t is the ratio of a polynomial of degrees L and M: $g(t) = \frac{p_0 + p_1 t + p_2 t^2 + \cdots + p_L t^L}{1 + q_1 t + q_2 t^2 + \cdots + q_M t^M} + O(t^{L+M+1})$. It provides approximately an analytically continued function of $g(t)$ outside the radius of convergence of $g(t)$. Thus the Pade approximation is expected to provide better results for small T than original plain series. For more detail, see for example, [25].

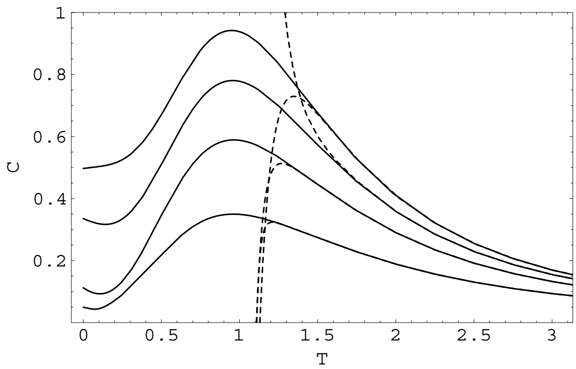

Figure 3. Temperature dependence of the high temperature expansion of the specific heat C for $J = 1$ at $h = 0$: broken lines denote plain series and smooth lines denote their Pade approximations for $s = 1, 2, 3, 4$ from the bottom to the top. Their orders are (n: plain series $O(1/T^n)$, Pade) = (52, [26, 26]), (21, [10, 10]), (15, [7, 7]), (12, [6, 6]) respectively. Their peak positions and peaks are (peak position, peak) = (0.961, 0.350), (0.963, 0.589), (0.956, 0.780), (0.949, 0.942) respectively. The case for $s = 1$ was calculated in [24].

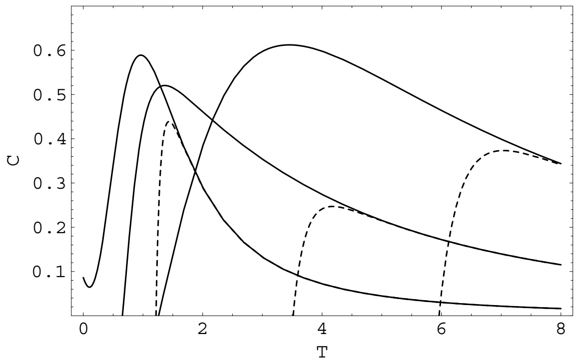

Figure 4. Temperature dependence of the high temperature expansion of the specific heat C for $J = 1, s = 2$: broken lines denote plain series and smooth lines denote their Pade approximations for $h = 0, 2, 4$ from the bottom to the top on the right-hand side. Their order is (n: plain series $O(1/T^n)$, Pade) = (17, [8, 8]). Their peak positions and peaks are (peak position, peak) = (0.964, 0.589), (1.365, 0.520), (3.452, 0.612), respectively.

5. Concluding remarks

In this paper, we have derived a NLIE with only *one* unknown function. This type of NLIE for higher spin Heisenberg model with *arbitrary* spin is derived for the first time. In particular, a remarkable connection between the NLIE and the quantum Jacobi–Trudi and Giambelli formula is first found.

Now that the NLIE is given, the next important task is to search for its solutions. The zeroth order of the high temperature expansion (4.1) is a solution of the Q-system. Thus this task is to incorporate the spectral parameter into a solution of the Q-system, namely to find a solution of the T-system. The solution of the Q-system is a kind of generalization of the hypergeometric function (cf [26]). Thus we expect that the final answer will be a further generalization of the hypergeometric function. If a hypergeometric series solution is found,

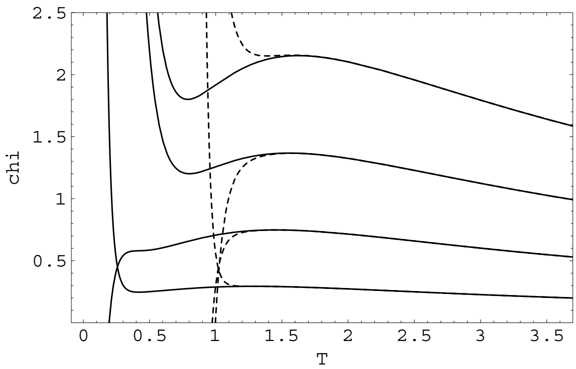

Figure 5. Temperature dependence of the high temperature expansion of the magnetic susceptibility χ for $J = 1$ at $h = 0$: broken lines denote plain series and smooth lines denote their Pade approximations for $s = 1, 2, 3, 4$ from the bottom to the top on the right-hand side. Their orders are (n: plain series $O(1/T^n)$, Pade) = (28, [13, 13]), (18, [8, 8]), (14, [7, 7]), (12, [6, 6]), respectively. Their first peak positions from the right and the peaks are (peak position, peak) = (1.282, 0.294), (1.454, 0.748), (1.555, 1.367), (1.624, 2.153), respectively. The case for $s = 1$ was calculated in [24].

one should consider an integral representation of it; then one may be able to treat the low temperature region where the plain series does not converge by an analytic continuation.

Finally, we note that we can also derive a NLIE similar to (3.9) for the row to row transfer matrix.

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